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MULTI-POINT HYBRID VIBRATION ABSORPTION IN FLEXIBLE STRUCTURES

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This study addresses the application of n hybrid vibration absorbers (HVAs) to a flexible structure. The HVAs assign tunable zeros and poles to a MIMO closed-loop flexible system and match it to a prototype transfer function with prescribed zeros and poles. Simulation results are presented to demonstrate the performance of the proposed design method.

1. INTRODUCTION

A dynamic vibration absorber (DVA) is a sprung–mass that was used for vibration absorption from as early as 1909 [1]. It is still an effective vibration control device [2, 3] nowadays, with different modifications and improvements. Some researchers adjust the absorption frequency by on-line tuning of the mass (inertia) or stiffness [4–6]. Others add an additional actuation force to a DVA to form a hybrid vibration absorber (HVA) [7–11]. The performance of a HVA depends on its actuation force. Different algorithms are available for HVA synthesis, ranging from neural networks [8], delayed resonator [9] to modal feedback controllers [10–14].

A possible HVA synthesis is to place closed-loop poles by modal feedback. While this is an effective control strategy for flexible structures, it could be improved in at least two aspects. First, most pole-placement controllers do not pay much attention to closed-loop zeros though zeros mean absorption band or valleys. Second, modal feedback requires exact mode functions to recover the modal states from sensor signals. In practice, exact knowledge of mode functions may not be conveniently available, or not accurate enough. It is therefore desirable to design HVAs to use sensor signals directly without recovering modal states. Such HVA schemes are called "simple HVA" here to distinguish from modal HVAs.

There are many interesting simple HVA schemes. A delayed resonator by Olgac and Holm-Hansen [9] tunes the attenuation frequency by adjusting a delayed feedback term. A double-resonance absorber by Burdisso and Heilnann [15] and a band-pass one by Filipovic and Schroder [16] present revolutionary ways to widen the attenuation valleys. Recently, a simple HVA was proposed [17] to place tunable zeros and poles for broadband absorption.

Unlike modal HVAs, a simple HVA usually absorbs vibration at a single point. Its analysis is based on a lumped-parameter model. This study addresses the application of n simple HVAs to flexible structures for vibration absorption in multiple points. Based on a distributed-parameter model, it can be shown that n simple HVAs can place zeros and poles to a flexible structure for vibration absorption in multiple points. The number of

HVAs (n) is independent of, some time much smaller than, the number of modes (m) to be attenuated. Feedback signals are measured at n coupling points, without using exact mode functions to recover modal states. It saves on-line computations, avoids modal error and requires fewer sensors. A simulation is conducted to demonstrate the absorption performance.

2. MATHEMATICAL MODEL

This study focuses on the design and application of n HVAs to a flexible structure for vibration absorption in n points. A simple example of the flexible structure would be a cantilever beam shown in Figure 1, where $w(x, t)$ represents displacement of the distributed-parameter beam while $\{w_j\}_{j=1}^{n}$ represent displacement of n lumped-parameter HVAs. It is assumed that the beam satisfies the Euler–Bernoulli hypothesis for displacement and Kelvin–Voigt damping hypothesis. All displacement variables are measured from the equilibrium states. The dynamic equations of the composite structure are given by

$$\rho\ddot{w} + C_d I\dot{w}'''' + EIw'''' = d(t)\delta(x - x_d) + \sum_{j=1}^{n} f_j\delta(x - x_j), \quad 0 < x < l, \tag{1a}$$

$$f_j = v_j[\dot{w}_j - \dot{w}(t, x_j)] + k_j[w_j - w(t, x_j)] + f_{aj}, \quad 1 \leqslant j \leqslant n, \tag{1b}$$

$$m_j\ddot{w}_j = f_j, \tag{1c}$$

plus a proper set of initial/boundary conditions. In equation (1a), $\delta(x - x_j)$ is a Dirac delta function and x_j the coupling co-ordinate of the jth HVA, E, ρ and C_d are the Young's modulus, linear mass density and damping coefficients of the beam, $d(t)$ represents a disturbance not drawn explicitly in Figure 1. The HVAs are sprung-masses each with an additional actuator. Equation (1b) models the coupling force between the flexible structure and the jth HVA where f_{aj} is a force synthesized by the actuator, m_j, v_j and k_j denote the corresponding mass, viscous friction and spring constants. A simple Newton's law in equation (1c) represents the dynamics of the jth HVA.

The vibration of this system is described by vector $\mathbf{w}^{\mathrm{T}} = [w(t, x), w_1, \ldots, w_n]$. According to Banks *et al.* [18], it is almost impossible to find a complete set of orthonormal mode functions to decompose w in modal space if the actuators are all shut-off ($f_{aj} = 0$ for $1 \leqslant j \leqslant n$). This is one of the difficulties of analyzing a composite system with discrete and continuous vibration bodies. The problem becomes more complicated if the actuators generate active forces f_{aj} for $1 \leqslant j \leqslant n$. Despite these difficulties, the classical modal theory still allows one to find a complete set of orthonormal mode functions $\{\phi_i(x)\}_{i=1}^{\infty}$ and

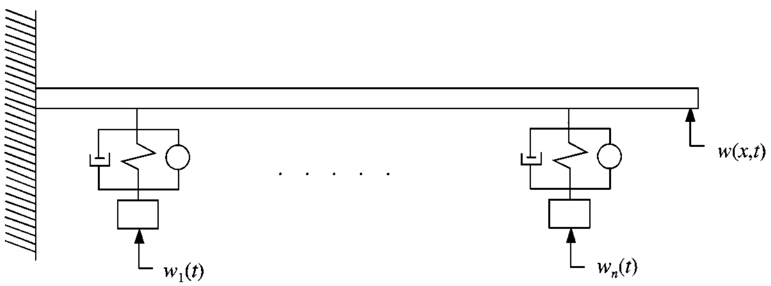

Figure 1. A cantilever beam coupled with n HVAs.

decompose $w(t, x)$ (instead of an entire vector $\mathbf{w}$) in modal space. Substituting $\{\phi_i(x)\}_{i=1}^{\infty}$ into equation (1a), one obtains

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = \phi_i(x_d)\,d + \sum_{j=1}^{n}\phi_i(x_j)f_j, \quad 1 \leqslant i < \infty, \tag{2a}$$

$$f_j = v_j[\dot{w}_j - \dot{w}(t, x_j)] + k_j[w_j - w(t, x_j)] + f_{aj}, \quad 1 \leqslant j \leqslant n, \tag{2b}$$

$$m_j\ddot{w}_j = -f_j, \tag{2c}$$

where ζ_i and ω_i represent the damping ratio and resonance frequency for the ith mode. The local modal decomposition is made possible by inner products

$$\eta_i(t) = \int_{x=0}^{l}\phi_i(x)w(t, x)\,\mathrm{d}x = \frac{EI}{\rho\omega_i^2}\int_{x=0}^{l}\phi_i(x)w''''(t, x)\,\mathrm{d}x,$$

$$\dot{\eta}_i(t) = \frac{C_d I}{2\zeta_i\omega_i\rho}\int_{x=0}^{l}\phi_i(x)\dot{w}''''(t, x)\,\mathrm{d}x \quad \text{and} \quad \int_{x=0}^{l}\phi_i(x)\phi_j(x)\,\mathrm{d}x = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

For a general flexible structure, the above inner products would be volume integrals over the continuous subsystem but still avoiding the discrete variables. This is a deliberate effort to avoid the extreme difficulty investigated by Banks *et al.* [18] when the inner products involve all displacement variables simultaneously. By avoiding the discrete variables in deriving equation (2a), one can still apply the classical modal theory without contradicting Banks *et al.* [18]. As a matter of fact, every mode co-ordinate $\eta_i(t)$ is strongly cross-coupled with all discrete variables $\{w_j\}_{j=1}^{n}$ via the coupling forces $\{f_j\}_{j=1}^{n}$, as indicated by equation (2a). While $\{\phi_i(x)\}_{i=1}^{\infty}$ is a complete set of orthonormal mode functions for continuous variable $w(t, x)$, it does not imply, nor depend on, the existence of orthonormal decomposition for the entire system vector $\mathbf{w}^{\mathrm{T}} = [w(t, x), w_1, \ldots, w_n]$. Derivation of equation (2a) is merely a different analytical approach without changing the validity of equation (1a).

While equation (1a) describes an 1-D flexible beam, the methodology presented here can be applied to a general 3-D flexible structure as well. From this point of view, equation (2a)–(2c) describe a set of general dynamic equations for a flexible structure after applying modal decomposition to the continuous subsystem while leaving the discrete variables unchanged. Of course, there are many other analysis methods in the literature. Some are more advanced in the modelling and analysis of flexible structures coupled with rigid bodies. The focus here, however, is how to synthesize the active coupling force f_{aj} and place closed-loop zeros and poles. While equation (2a) seems to depend on the existence of mode functions for the continuous subsystem, the design procedures to be developed here do not use these mode functions explicitly. It will be made clear that equations (2a)–(2c) are used to establish a useful property of the transfer functions. In reality, these transfer functions are identified from measurement data instead of being derived analytically. The next section will present a HVA design procedure that depends on the measured transfer functions instead of model (2a)–(2c). Meanwhile this section addresses the property of transfer functions. One may express equation (2a) in the Laplace transform domain as

$$\eta_i = \frac{\phi_i(x_d)\,d + \sum_{j=1}^{n}\phi_i(x_j)f_j}{s^2 + 2\zeta_i\omega_i s + \omega_i^2} \quad \text{for } 1 \leqslant i \leqslant \infty.$$

It implies a spatial-temporal expression for the continuous subsystem

$$w(x_k, s) \approx \sum_{i=1}^{m} \phi_i(x_k) \frac{\phi_i(x_d)\, d + \sum_{j=1}^{n} \phi_i(x_j) f_j}{s^2 + 2\zeta_i \omega_i s + \omega_i^2} \quad \text{for } 1 \leqslant k \leqslant n. \tag{3}$$

The truncation to the first m modes is a well-accepted engineering approximation. A vector–matrix form is now available for equation (3) as

$$w_p = \Phi^{\mathrm{T}} \left[\frac{1}{s^2 + 2\zeta_i \omega_i s + \omega_i^2} \right] (\phi_d d + \Phi f), \tag{4}$$

where $w_p^{\mathrm{T}}(s) = [w(x_1, s)\ w(x_2, s) \cdots w(x_n, s)]$ and $f^{\mathrm{T}}(s) = [f_1(s)\ f_2(s) \cdots f_n(s)]$ are the displacements of the flexible structure and the coupling forces between the HVAs and the flexible structure; $\phi_d^{\mathrm{T}} = [\phi_1(x_d)\ \phi_2(x_d) \cdots \phi_n(x_d)]$ describes spatial magnitudes of the disturbance; $[1/(s^2 + 2\zeta_i\omega_i s + \omega_i^2)]$ is a diagonal matrix whose ith diagonal element is

$$\frac{1}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}, \quad \text{and} \quad \Phi^{\mathrm{T}} = \begin{bmatrix} \phi_1(x_1) & \phi_2(x_1) & \phi_3(x_1) & \cdots & \phi_m(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \phi_3(x_2) & \cdots & \phi_m(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_n) & \phi_2(x_n) & \phi_3(x_n) & \cdots & \phi_m(x_n) \end{bmatrix} \text{ is the mode matrix.}$$

All temporal signals in equation (4) are expressed in Laplace transform domain. For HVA design, one may introduce the transfer functions

$$G_d(s) = \Phi^{\mathrm{T}} \left[\frac{1}{s^2 + 2\zeta_i\omega_i s + \omega_i^2} \right] \phi_d \quad \text{and} \quad G_f(s) = \Phi^{\mathrm{T}} \left[\frac{1}{s^2 + 2\zeta_i\omega_i s + \omega_i^2} \right] \Phi. \tag{5}$$

These transfer functions render a concise expression of equation (4) as

$$w_p = G_d(s) d + G_f(s) f = \frac{1}{D_p(s)} \{ N_d(s) d + N_f(s) f \}, \tag{6}$$

where $D_p(s) = \prod_{i=1}^{m} (s^2 + 2\zeta_i\omega_i s + \omega_i^2)$ is the system characteristic polynomial, and $N_d(s)$ and $N_f(s)$ are the numerators of $G_d(s)$ and $G_f(s)$ respectively. Similar to Huang and Fuller [19], $D_p(s)$ and $N_f(s)$ are assumed to be available by analysis, measurement or off-line identification.

There are many available pole-placement controllers for active damping of flexible structures like equation (6). Some use a lumped-parameter system with a number of sprung-masses to approximate a distributed-parameter system. For those HVA designs that are based on the distributed-parameter models, a modal state vector $[\eta_1\ \eta_2\ \cdots\ \eta_m]^{\mathrm{T}}$ is usually required whose components must be computed by modal filters

$$\eta_i(t) = \int_L \phi_i(x) w(x, t)\, \mathrm{d}x \approx \sum_{k=1}^{K} \phi_i(x_k) w(x_k, t)\, \delta x, \quad 1 \leqslant i \leqslant m,$$

where $\{w(x_k, t)\}_{k=1}^{K}$ are displacements measured at co-ordinates $\{x_k\}_{k=1}^{K}$. To achieve a reasonably accurate approximation, the number of sensors (K) must be sufficiently large or at least equal to m. That requires a large number of sensors. Generally, modal filters require exact knowledge of mode functions, which may not be conveniently available or not accurate enough in practice. The present approach, on the other hand, does not use mode functions explicitly, though it needs equation (5) to show a useful property of the transfer functions and equation (5) must be derived from equation (2a). In real implementations, there may be practical restrictions on the total number of sensors. Such problems could cause inevitable errors in the modal vector. Using transfer functions $G_d(s)$ and $G_f(s)$ and direct feedback $\{w(x_k, t)\}_{k=1}^{K}$, the proposed HVA scheme saves on-line computations, reduces sensors and avoids potential modal errors.

3. HVA DESIGN

Assume, without loss of generality, that $m_j = m_a$, $v_j = v_a$ and $k_j = k_a$ for $1 \leqslant j \leqslant n$. Then a relatively simple design scheme is to synthesize f_{aj} individually for each HVA. Expressing equation (2b) in Laplace transform domain, one can write

$$f = (sv_a + k_a)I(w_a - w_p) - A(s)w_p, \tag{7}$$

where $w_a^{\mathrm{T}} = [w_1 \;\; w_2 \;\; \cdots \;\; w_n]$, $f_a = -A(s)w_p$, $f_a^{\mathrm{T}} = [f_{a1} \; f_{a2} \cdots f_{an}]$, I is an $n \times n$ identity matrix and $A(s)$ is an $n \times n$ HVA transfer function to be designed in this section. The spring–dampers contribute the passive part by $(sv_a + k_a)I(w_a - w)$ and the actuators contribute the active part by $f_a = -A(s)w_p$.

Equation (2c) has a vector form $w_a = (-1/m_a s^2)I_f$, where I is an $n \times n$ identity matrix. It can be substituted into equation (7) to eliminate w_a. As the result one obtains

$$f = -\frac{s^2}{B(s)}\{(sv_a + k_a)I + A(s)\}\, w_p, \tag{8}$$

where $B(s) = s^2 + s\sigma + \kappa$, $\sigma = v_a/m_a$ and $\kappa = k_a/m_a$. A combination of equations (8) and (6) results in $w_p = G_c(s)d$. Here $G_c(s)$ is the closed-loop transfer function of the primary system given by

$$G_c(s) = \{P(s) + s^2 N_f(s) A(s)\}^{-1} B(s) N_d(s), \tag{9}$$

where $P(s) = B(s)D_p(s)I + s^2(sv_a + k_a)N_f(s)$ is a constant matrix polynomial of degree $2m + 2$. Its coefficient matrices are denoted by $\{P_i\}_{i=0}^{2m+2}$. Particularly, $P_0 = I$ since $P(s)$ is monic.

If the actuators are shut-off ($A(s) = 0$), then equation (9) becomes $G_c(s) = P^{-1}(s)B(s)N_d(s)$ where $B(s)$ contributes a pair of closed-loop zeros for vibration absorption. That is the effect of passive DVAs. The proposed HVAs select $\kappa = k_a/m_a$ exactly the same way as passive DVAs to tune an absorption frequency. A drawback of the DVAs, however, is the possible detrimental effect to $P(s) = B(s)D_p(s)I + s^2(sv_a + k_a)N_f(s)$ whose roots are altered by the DVAs. When applied to a lumped-parameter system, a DVA could cause closed-loop peaks [2, 3] in other frequencies. For distributed-parameter systems, the effect of a DVA is more difficult to predict [19]. The problem becomes more complicated when multiple DVAs are involved. The main objective of the HVAs is to shift the roots of $P(s)$ away from the imaginary axis and place more closed-loop zeros to attenuate vibration in other frequencies. Such a concept has been applied to lumped-parameter system successfully [17] with a single HVA. This study addresses the application of n HVAs to a flexible system to absorb vibration in multiple points.

In view of equation (5), one can see

$$Re[(\mathrm{j}\omega + \varepsilon)G_f(\mathrm{j}\omega)] = \Phi^{\mathrm{T}}\left[\frac{\varepsilon\omega_i^2 + (2\zeta_i\omega_i - \varepsilon)\omega^2}{(\omega_i^2 - \omega^2)^2 + 4\zeta_i^2\omega_i^2\omega^2}\right]\Phi > 0$$

for all ω when $\min\{2\zeta_i\omega_i\} > \varepsilon > 0$. It implies a strictly positive real $(s+\varepsilon)G_f(s)$ for a distributed-parameter model with m modes. Therefore, $N_f^{-1}(s)$ is a stable transfer function matrix. The actuator transfer function is given by

$$A(s) = A_1 s + \frac{1}{C(s)} N_f^{-1}(s)H(s) + \frac{1}{s} A_3 + \frac{1}{s^2} A_4, \tag{10}$$

where A_1, A_3 and A_4 are $n \times n$ coefficient matrices, $C(s) = s^l + \sum_{i=1}^{l} c_i s^{l-i}$ is a scalar polynomial with l prescribed zeros and $H(s) = \sum_{i=0}^{l+2m-2} H_i s^{l+2m-2-i}$ is a $n \times n$ polynomial matrix of $l + 2m - 2$. Combining equations (9) and (10), one obtains

$$G_c(s) = \{M(s) + R(s)(A_1 s^3 + A_3 s + A_4) + s^2 H(s)\}^{-1} C(s)B(s)N_d(s), \tag{11}$$

where

$$M(s) = C(s)P(s) = \sum_{i=0}^{l+2m+2} M_i s^{l+2m+2-i} \quad \text{and} \quad R(s) = C(s)N_f(s) = \sum_{i=0}^{l+2m-2} R_i s^{l+2m-2-i}.$$

The leading coefficient of $M(s)$ is an identity I since both $P(s)$ and $C(s)$ are monic.

With $C(s)B(s)$ being a closed-loop multiplier, the HVAs place $l + 2$ tunable zeros in the closed-loop transfer function. To place $l + 2m + 2$ closed-loop poles to equation (11), the design method needs a scalar prototype denominator.

$$E(s) = s^{l+2m+2} + e_1 s^{l+2m+1} + e_2 s^{l+2m} \cdots + e_{l+2m} s^2 + e_{l+2m+1} s + e_{l+2m+2},$$

that is constructed from selectable poles. The remaining task is to choose A_1, A_3, A_4 and $H(s)$, such that

$$E(s)I = M(s) + R(s)(A_1 s^3 + A_3 s + A_4) + s^2 H(s). \tag{12}$$

Attention is now directed to matrix polynomial product

$$\begin{aligned} R(s)(A_1 s^3 + A_4) = {} & R_0 A_1 s^{l+2m+1} + R_1 A_1 s^{l+2m} + R_2 A_1 s^{l+2m-1} + (R_3 A_1 + A_4) s^{l+2m-2} \\ & + (R_4 A_1 + R_1 A_4) s^{l+2m-3} \cdots + (R_{k+3} A_1 + R_k A_4) s^{l+2m-2-k} \cdots \\ & + (R_{l+2m-2} A_1 + R_{l+2m-5} A_4) s^3 + R_{l+2m-4} A_4 s^2 + R_{l+2m-3} A_4 s \\ & + R_{l+2m-2} A_4 \end{aligned}$$

whose first and last terms are tunable by A_1 and A_4 respectively. Setting $A_1 = R_0^{-1}(e_1 I - M_1)$ and $A_4 = R_{l+2m-2}^{-1}(e_{l+2m+2} I - M_{l+2m+2})$, respectively, one achieves

$$M(s) + R(s)(A_1 s^3 + A_4) = (s^{l+2m+2} + e_1 s^{l+2m+1} + e_{l+2m+2})I + s \sum_{i=0}^{l+2m-1} Q_i s^{l+2m-1-i} \tag{13}$$

that matches at least three terms of $E(s)I$. The coefficient matrices $\{Q_i\}_{i=1}^{l+2m}$ do not necessarily match the corresponding terms in $E(s)I$. These will be addressed step by step. From equation (13), it is not difficult to see

$$\begin{aligned} & M(s) + R(s)(A_1 s^3 + A_3 s + A_4) \\ & \quad = (s^{l+2m+2} + e_1 s^{l+2m+1} + e_{l+2m+2})I + (R_{l+2m-2} A_3 + Q_{l+2m-1}) s + s^2 \sum_{i=0}^{l+2m-2} F_i s^{l+2m-2-i}. \end{aligned}$$

A selection of $A_3 = R^{-1}_{l+2m-2}(e_{l+2m+1}I - Q_{l+2m-1})$ leads to

$$M(s) + R(s)(A_1 s^3 + A_3 s + A_4)$$
$$= (s^{l+2m+2} + e_1 s^{l+2m+1} + e_{l+2m+1}s + e_{l+2m+2})I + s^2 \sum_{i=0}^{l+2m-2} F_i s^{l+2m-1}. \quad (14)$$

Now that $F(s) = \sum_{i=0}^{l+2m-2} F_i s^{l+2m-2-i}$ and $H(s)$ have the same degree, $s^2[F(s) + H(s)]$ is completely tunable. Equation (12) holds if one substitutes equation (14) and $H_i = e_{i+2}I - F_i$ for $0 \leqslant i \leqslant l + 2m - 2$.

The objective of zero/pole placement is now achieved for multi-point vibration absorption in a flexible structure. The assumption of the proposed scheme is availability of $N_f(s)$ and $D_p(s)$, which is the same as the assumption of Huang and Fuller [19]. If $N_d(s)$ is minimum phase and available as well, then one can select an $E(s)$ to cancel $N_d(s)$ in the closed-loop transfer function. In any of the cases, all poles and some zeros are tunable by a designer by prescribing $B(s)$, $C(s)$ and $E(s)$ respectively. Roots of these scalar polynomials are restricted in the negative half of the complex plane to ensure a stable closed loop. Roots of $B(s)$ and $C(s)$, however, can be close to the imaginary axis to form sharp attenuation valleys or stop bands.

4. IMPLEMENTATION

This section addresses several issues relevant to the implementation of the proposed scheme. The most important one is the existence of R_0^{-1} and R^{-1}_{l+2m-2}. These matrices determine A_1, A_3 and A_4 respectively. Since $R(s) = C(s)N_f(s)$, $R_0 = c_0 N_{f0}$ and $R_{l+2m-2} = c_1 N_{f2m-2}$. Scalar polynomial $C(s) = \prod_{i=1}^{1/2}(s^2 + \beta_i^2)$ places zero to the closed loop with $c_0 = 1$ and $c_l = \prod_{i=1}^{1/2} \beta_i^2 > 0$. The existence of R_0^{-1} and R^{-1}_{l+2m-2} depends on the singularities of N_{f0} and N_{f2m-2}. Equation (5) implies

$$G_f(0) = \frac{1}{\prod_{i=1}^m \omega_i^2} N_{f2m-2} = \Phi^{\mathrm{T}} \left[\frac{1}{\omega_i^2}\right] \Phi > 0$$

and hence the existence of R^{-1}_{l+2m-2}. Similarly, $\lim_{s\to\infty}\{s^2 G_f(s)\} = N_{f0} = \Phi^{\mathrm{T}}\Phi > 0$ can be derived from equation (5) to establish the existence of R_0^{-1}. This paves the way for the computation of A_1, A_3 and A_4.

Another issue is the matrix filter $(1/C(s))\ N_f^{-1}(s)H(s) = R^{-1}(s)H(s)$ with a high order $l + 2m - 2$. This problem may be eased by a state-space implementation that reduces a high order filter to a first order vector integral. Expressed in state space, $y(s) = R^{-1}(s)H(s)w(s)$ has the form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\bar{R}_{l+2m-2} \\ I & 0 & \cdots & 0 & -\bar{R}_{l+2m-3} \\ 0 & I & \cdots & 0 & -\bar{R}_{l+2m-4} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & -\bar{R}_1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} H_{l+2m-2} - \bar{R}_{l+2m-2}H_0 \\ H_{l+2m-3} - \bar{R}_{l+2m-3}H_0 \\ H_{l+2m-4} - \bar{R}_{l+2m-4}H_0 \\ \vdots \\ H_1 - \bar{R}_1 H_0 \end{bmatrix} w(t),$$

where $\{\bar{R}_i = R_0^{-1}R_i\}_{i=1}^{l+2m-2}$ and $y(t) = R_0^{-1}\{z_n(t) + H_0 w(t)\}$. Available electronics are generally much faster than the response of physical flexible structures. The above first order integral can be very fast and accurate. The technique has been applied to switch capacitor circuits to implement high order analog filters successfully. It is readily applicable to HVAs as well.

A third concern is the double integral in equation (10). This can be eliminated by setting $A_4 = 0$ if one feels uncomfortable with double integral in $A(s)$. Recall that

$$M(s) = C(s)P(s) = C(s)\{B(s)D_p(s)I + s^2(sv_a + k_a)N_f(s)\}$$
$$= \prod_{i=1}^{1/2}(s^2+\beta_i^2)(s^2+\kappa)\prod_{i=1}^{m}(s^2+\omega_i^2)I + s^2C(s)(sv_a+k_a)N_f(s), \quad (15)$$

where $C(s) = \prod_{i=1}^{1/2}(s^2+\beta_i^2)$, $B(s) = (s^2+\kappa)$ and $D_p(s) = \prod_{i=1}^{m}(s^2+\omega_i^2)$ exaggerate the under damping of these polynomials in real implementations. Since $s^2C(s)(sv_a+k_a)N_f(s)$ only contribute to s^2 or higher, the zero order coefficient of $M(s)$ is a scalar multiplied to an identity: $M_{l+2m+2} = \kappa\prod_{i=1}^{1/2}\beta_i^2\prod_{i=1}^{m}\omega_i^2 I = m_{l+2m+2}I$. Here $\sqrt{\kappa}$, $\{\beta_i\}_{i=1}^{1/2}$ are prescribed roots of $B(s)$ and $C(s)$ to be placed as closed-loop zeros. $D_p(s)$ and $\{\omega_i\}_{i=1}^{m}$ are available by assumption. Without losing generality, one may choose

$$E(s) = \prod_{i=1}^{1/2}(s^2+2\beta_i s+\beta_i^2)(s^2+2\sqrt{\kappa}s+k)\prod_{i=1}^{m}(s^2+2\omega_i s+\omega_i^2)$$

whose zero order coefficient is $e_{l+2m+2} = \kappa\prod_{i=1}^{1/2}\beta_i^2\prod_{i=1}^{m}\omega_i^2 I = m_{l+2m+2}$ for whatever damping ratio introduced to $E(s)$. Therefore, equation (13) holds when $A_4 = R_{l+2m-2}^{-1}(e_{l+2m+2} - m_{l+2m+2})I = 0$. One can follow the procedures of Section 3 to place the prescribed zeros and poles without the double integral.

Section 3 assumes identical spring–mass-viscous coefficients to avoid distraction in deriving $A(s)$. There is, in fact, no problem applying n HVAs with different spring–mass-viscous coefficients. Only the coupling force vector becomes

$$f = -\left[\frac{s^2}{B_i(s)}\right]\{[sv_i + k_i]I + A(s)\}w_p, \quad (16)$$

where $[s^2/B_i(s)]$ and $[sv_i + k_i]$ are diagonal matrices with diagonal elements $s^2/B_i(s)$ and $sv_i + k_i$ respectively; $B_i(s) = s^2 + s\sigma_i + \kappa_i$ represents resonance of the ith sprung-mass with coefficients $\sigma_i = v_i/m_i$ and $\kappa_i = k_i/m_i$. Substituting equation (16) into equation (6), the closed-loop transfer function of the flexible system can be derived as

$$G_c(s) = \{\bar{P}(s) + s^2N_f(s)[G_i(s)]A(s)\}^{-1}\bar{B}(s)N_d(s), \quad (17)$$

where $\bar{P}(s) = \bar{B}(s)D_p(s)I + s^2N_f(s)[G_i(s)(sv_i + k_i)]$ is a matrix polynomial of degree $2m + 2n$; $\bar{B}(s) = \prod_{i=1}^{n}B_i(s)$; $[G_i(s)(sv_i + k_i)]$ and $[G_i(s)]$ are diagonal matrices with diagonal elements $G_i(s) = \bar{B}(s)/B_i(s)$ for $1 \leqslant i \leqslant n$. The differences between equations (9) and (17) are $P(s)$ versus $\bar{P}(s)$ and $B(s)$ versus $\bar{B}(s)$. Instead of absorbing vibration at a single frequency, $\bar{B}(s)$ allows the HVAs to absorb vibration at n-frequencies in n points. The design of $A(s)$ is very similar to Section 3 with $N_f^{-1}(s)$ replaced by $[G_i(s)]^{-1}N_f^{-1}(s)$.

In the absence of A_4, equation (10) becomes sort of a PID controller with its proportional gain replaced by a high order matrix filter $(1/C(s))N_f^{-1}(s)H(s)$, or $1/C(s)G_i(s)]^{-1}N_f^{-1}(s)H(s)$ if the HVA's have different spring constants. The HVA transfer function may be further simplified by selecting $C(s) = 1$. In that case, the zeros of $B(s)$ or $\bar{B}(s)$ absorb vibration and $A(s)$ damps the closed-loop poles. A simulation example is presented in the next section to demonstrate the performance.

5. SIMULATION RESULTS

The proposed scheme is verified by a simulation test. Only $n = 1$ HVA is applied to a flexible 1-D cantilever beam. With $n = 1$ sensor collocating with a HVA, it is not possible to recover $m > 1$ modal states by any modal HVAs. Yet the proposed HVA is able to damp $m > 1$ modes. The simulation also demonstrates that the procedures of Section 3 are applicable to a special case with $n = 1$. The only modification is to change the matrices into scalars. The flexible beam has a model

$$EL\frac{\partial^4 w}{\partial x^4} + \rho\frac{\partial^2 w}{\partial t^2} = \delta(x - 0.2L)d + \delta(x - 0{\cdot}8L)f,$$

where w is the vibration displacement, x and t are spatial and temporal variables, and L denotes the beam length. The disturbance and the absorber act at $x_d = 0{\cdot}2L$ and $x_f = 0{\cdot}8L$ respectively. The feedback sensor collocates with the HVA at $x_f = 0{\cdot}8L$. Mode functions (not normalized) are analytically available as

$$\varphi_i(x) = [\cosh(\lambda_i x) - \cos(\lambda_i x)] - \mu_i[\sinh(\lambda_i x) - \sin(\lambda_i x)],$$

where $\mu_i = [\cos(\lambda_i L) + \cosh(\lambda_i L)]/[\sin(\lambda_i L) + \sinh(\lambda_i L)]$ is a constant depending on λ_i, the ith root of frequency equation $\cos(\lambda_i L)\cosh(\lambda_i L) + 1 = 0$. Eigenvalue λ_i also determines the ith mode frequency $\omega_i = (\lambda_i L)^2\sqrt{EI/\rho L^4}$. Without loss of generality, the simulation assumes $\sqrt{EI/\rho L^4} = 1$, and damping ratio of all modes to be 0·005.

A truncated model (to the first 5 modes) is adopted in the simulation, since the fifth and the sixth mode peaks are significantly lower than the first four peaks when the beam is excited by the disturbance as shown in Figure 2 (without the HVA). Hence, the truncation is reasonable. The mode frequencies are $\omega_1 = 15{\cdot}4118$, $\omega_2 = 49{\cdot}9648$, $\omega_3 = 104{\cdot}2477$, $\omega_4 = 178{\cdot}2697$ and $\omega_5 = 272{\cdot}0309$ (rad/s) respectively.

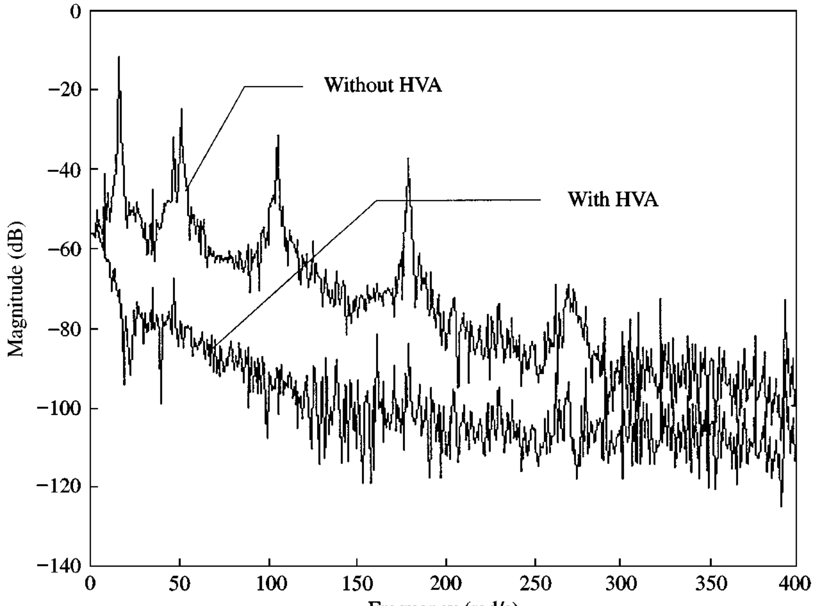

Figure 2. Simulated performance of the proposed HVA.

The simulated HVA synthesis the actuation force by $A(s) = a_1 s + H(s)/N_f(s) + a_3/s$ where $A(s)$ is sort of a PID controller transfer function with $C(s) = 1$. The closed-loop transfer function of the flexible system is derived as

$$\frac{w_p}{d} = \frac{N_d(s)B(s)}{M(s) + (a_1 s^3 + a_3 s)N_f(s) + s^2 H(s)}, \tag{18}$$

where $B(s) = s^2 + 0{\cdot}01s + 400$ contributes an attenuate valley at 20 rad/s while

$$M(s) = B(s)D_p(s) + s^2 N_f(s)(v_a s + k_a) = s^{12} + m_1 s^{11} + m_3 s^{10} + \cdots + m_{10}s^2 + m_{11}s + m_{12}.$$

It is not difficult to see $m_{12} = \kappa \prod_{i=1}^{5} \omega_i^2$. The prototype denominator of closed-loop transfer function is selected as

$$E(s) = (s^2 + 40s + 400) \prod_{i=1}^{5} (s^2 + 2\omega_i s + \omega_i^2),$$

which implies $e_{12} = m_{12}$. This avoids the double integral a_4/s^2 while satisfying equation (12). All roots of $E(s)$ are critically damped to remove the mode peaks. The choice of a_1, a_3 and $H(s)$ follows the procedures of Section 3 to match the denominator of equation (18) to $E(s)$. The matrix coefficients of Section 3 are simplified to scalars here to achieve the same objective.

The simulated performance of the HVA is shown in Figure 2. The closed-loop zeros and poles contribute positively to absorb vibration in a wide frequency range, though only one pair of closed-loop zeros is assigned at 20 rad/s. The scheme is proposed to apply $n = 2$ or 3 HVAs to a flexible structure to attenuate $m = 5$ or more modes. It requires fewer sensors, saves on-line computations and avoids possible modal errors when compared with modal feedback HVAs. Other simple HVAs are not able to achieve such an objective.

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